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Improved hard-thermal-loop effective action for hot QED and QCD

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Abstract

The conventional results for hard thermal loops, which are the building blocks of resummed perturbation theory in thermal field theories, have collinear singularities when external momenta are light-like. It is shown that by taking into account asymptotic thermal masses these singularities are removed. The thus improved hard thermal loops can be summarized by compact gauge-invariant effective actions, generalizing the ones found by Taylor and Wong, and by Braaten and Pisarski.

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1 Introduction

It is a typical phenomenon in thermal field theory that the conventional perturbative loop expansion breaks down because of infrared singularities which are usually cured by the generation of thermal masses[1]. In the particularly simple case of a massless $\lambda\phi^4$-theory, the leading temperature contribution to the self-energy consists of a simple mass term with $m = \sqrt{\lambda}T$. An improved perturbation theory requires that this thermal mass be treated on a par with the tree-level inverse propagator whenever the momentum of the latter becomes comparable to (or smaller than) the former. A consistent way of doing so is to add the thermal mass term to the bare Lagrangian and to subtract it again through counterterms at higher loop orders.

In general, however, the thermal self-energy is more complicated than a constant mass squared and it may also be the case that higher vertex functions are equally important. This is the case in QED and QCD, for which a resummed perturbation theory has been developed by Braaten and Pisarski[2]. The leading temperature contributions to be resummed have been obtained by Frenkel and Taylor[3] and independently by the former, who have coined the term "hard thermal loops". These hard thermal loops (HTL) turned out to satisfy ghost-free Ward identities[4,5]. Manifestly gauge invariant as well as rather compact expressions for the effective Lagrangians which summarize them were found in Refs. [6–9].

This resummation program has been applied successfully to a number of problems (see e.g. the forthcoming review in Ref. [10]). Occasionally, however, it turned out that the thermal masses contained in the HTL effective action were insufficient to screen all infrared singularities. For example, through quasiparticle mass-shell singularities, the damping rates of plasma excitations with nonzero momentum as well as the next-to-leading order screening corrections are blown up by unscreened magnetostatic modes [11]. However, assuming the existence of some effective infrared cutoff at the (unfortunately entirely nonperturbative) magnetic mass scale at least allows one to obtain a leading logarithmic correction.

Another shortcoming of the conventional HTL resummation has been revealed recently in the attempt to calculate the production rate of soft real photons from a quark-gluon plasma. In the case of hard real photons, a complete leading order calculation was carried out in Ref. [12], where it was found that the mass singularities of the quarks are screened by HTL corrections. But with soft real photons, the hard thermal loops themselves introduced uncancelled collinear singularities [13]. We shall concentrate on this latter type of singularities in what follows.

In a quite different context a similar difficulty was encountered in Ref. [14]. In the simpler gauge theory of scalar electrodynamics, the complete next-to-leading order dispersion laws of photonic excitations could be obtained in a resummed one-loop calculation. However, the perturbative result for the longitudinal branch turned out to become unreliable as the light cone was approached with increasing plasmon momentum. The reason for this breakdown was again a collinear singularity of a HTL diagram. There it was found that it can be removed by a further resummation.

In this paper, we extend this strategy to the case of QED and QCD and we shall show that a manifestly gauge invariant HTL effective action can be found which is completely free from collinear singularities. It improves upon the effective action of Refs. [6–8] with which it coincides for external momenta that are sufficiently far from the light-cone.

In the next section we argue that the hard propagators have to be dressed by asymptotic thermal masses whenever collinear singularities make the hard thermal loops themselves sensitive to such higher-order corrections. In sect. 3 we carry out a resummation of the asymptotic thermal masses first for the case of purely gluonic QCD, and in sect. 4 when fermions are included. The analytic structures obtained differ, but in each case the improved hard thermal loops can be summarized by a gauge invariant effective action. Sect. 5 contains our conclusions and gives an outlook to potential applications.

2 Hard thermal loops in the vicinity of the light-cone

Consider the 00-component of the polarization tensor (i.e. self-energy) at leading order as given by its hard thermal loop. It reads universally [15]

$$\begin{aligned}\Pi_{00}(Q) &= 4e^2 \int \frac{d^3p}{(2\pi)^3}\, n(p) \left\{ 1 - \frac{Q_0}{Q_0 - \mathbf{pq}/p} \right\} \\ &= 3m^2 \left(1 - \frac{Q_0}{2q} \ln \frac{Q_0 + q}{Q_0 - q} \right) \end{aligned} \tag{1}$$

with $m = eT/3$ and n is the Bose function, where e is either the coupling constant of QED or

$$e = g\sqrt{N + N_f/2} \tag{2}$$

for SU(N) with N_f fermions.

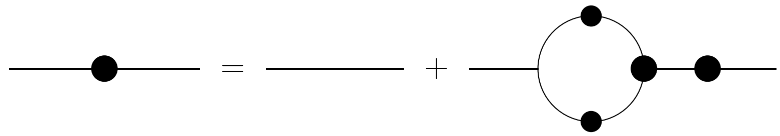

Fig. 1. Prototypical Schwinger-Dyson equation — for simplicity for a ϕ^3-theory. Internal propagators are always dressed, whereas vertices appear in bare and dressed versions.

Close to the light-cone Π_{00} is found to diverge logarithmically,

$$\Pi_{00} \sim -m^2 \ln \frac{1}{\varepsilon} \quad \text{with} \quad \varepsilon^2 = Q^2/q^2 \,. \tag{3}$$

The origin of this divergence is a collinear singularity in the loop integral (1) at $\mathbf{pq} = pq$ when $Q_0 = q$.

The next-to-leading order contribution for soft momenta $Q \equiv (Q_0, \mathbf{q})$, $Q_0, \mathbf{q} \sim eT$ is given by resummed one-loop diagrams. In both scalar electrodynamics [14] and QCD [16], it turns out to diverge like

$$\delta\Pi_{00} \sim m^2 \frac{e}{\varepsilon} \tag{4}$$

when approaching the light-cone. Evidently, the resummed perturbative series breaks down for $\varepsilon \lesssim e$, for then $\delta\Pi_{00} \gtrsim \Pi_{00}$.

In the case of scalar electrodynamics (massless and without scalar self-interactions) it is relatively easy to analyse this problem [14]. The collinear singularity in (3) is brought about by the masslessness of the internal scalar particles. The even stronger singularity in (4) is generated by a premature restriction to soft loop momenta. The scalar propagators are dressed and therefore massive, but one has to subtract off the contributions already covered by the bare HTL diagram. This again involves massless scalar propagators, which become large at the light-cone irrespective of the momentum scale. Thus (4) reveals a latent UV-divergence.

However, from the Schwinger-Dyson equations for the various Green functions (Fig. 1) it is clear that in the full theory the scalar lines are always massive with mass squared $m^2 = \frac{1}{4}e^2T^2(1 + O(e))$. The collinear singularities of Π_{00} at the light-cone, which are produced by scalar particles, should therefore be spurious.

Indeed, keeping the thermal masses also for the hard scalar propagators yields a finite result for the HTL contribution to Π_{00} where ε in (3) gets replaced by e. The increasingly singular contributions (3,4) are thus seen to get cancelled

by higher loop diagrams corresponding to a hard scalar loop where scalar self-energy diagrams are inserted repeatedly. Away from the light-cone, such insertions are suppressed by powers of e^2, but they cease to be so when $\varepsilon \lesssim e$.

Resumming thermal masses already at the stage where the hard thermal loops are being calculated raises the question about the systematics of this procedure. For this it is useful to think in terms of a renormalization group approach [17]. The conventional resummation program of Braaten and Pisarski can be understood as a two-step procedure. First an arbitrary scale Λ with $T \gg \Lambda \gg eT$ is introduced which divides all momenta and energies in hard and soft, and an effective theory is built from integrating out all the hard modes. To leading order, the result is independent of the actual value of Λ, and is given by the HTL effective action. In a second step, Green functions with external soft momenta are calculated using this effective action with soft loop momenta. The degree to which a perturbative expansion based on the effective theory can possibly make sense is limited by the accurateness of the effective action. Higher loop orders of the former require sufficiently high loop orders of the latter.

In the above example it became apparent that for soft momenta with $\varepsilon \lesssim e$ the effective action receives contributions from hard diagrams with arbitrarily high loop orders. In order to sum them systematically one would have to solve the Schwinger-Dyson equations, which are an infinite set of coupled equations. However, the present singularities are caused primarily by the masslessness of the bare propagators, whereas the structure of the Schwinger-Dyson equations is such that only full propagators enter (in contrast to vertex functions which appear in both bare and dressed forms). As long as it is sufficient to sum only self-energy insertions to produce massive propagators, there is no particular problem with overcounting.

In fact, also in the more complicated gauge theories like spinor electrodynamics and QCD the physical degrees of freedom retain thermal masses for momenta $p \gg eT$. In the case of gauge bosons, there is a transverse and a longitudinal branch of quasi-particle mass-shells, but the longitudinal branch rapidly dies out with increasing momentum. In the HTL approximation, the longitudinal plasmons approach the light-cone, but in doing so the corresponding residue vanishes exponentially [18]. The transverse mode, on the other hand, tends toward an effective asymptotic mass

$$m_\infty^2 \equiv \Pi_t(Q^2 = 0) = \frac{e^2T^2}{6} + O(e^2 T \Lambda) \tag{5}$$

and residue 1. (Recall that in QCD e is defined by (2).) Likewise, the dispersion laws for ultrarelativistic fermions have a collective branch (the "plasmino" with a flipped relation between helicity and chirality) which dies out with in-

creasing momentum and a normal one that remains, again with an asymptotic mass proportional to $e^2T^2 + O(e^2T\Lambda)$ (without spoiling chirality) [19].

The HTL effective action is generated entirely by the physical degrees of freedom of the bare theory, which is particularly evident when the Coulomb gauge is used, where only transverse modes are heated. [3] To leading order, these contribute only as real particles living on their mass-shell, which is the light-cone $Q^2 = 0$. Correspondingly, the collinear singularities present in conventional HTL diagrams with external light-like momenta are all removed when the above asymptotic thermal masses for transverse gauge bosons and any other ultrarelativistic particles are included.

It is an important feature of the asymptotic thermal masses that the result (5) as well as the corresponding one for fermions (see below) does not depend on the condition $Q_0, q \ll T$ which is otherwise essential for the derivation of the HTL results. It holds for arbitrary momentum as shown in Eq. (A.21) of Ref. [21]. Moreover, the definition (5) is itself not sensitive to a soft modification of the hard propagators, because Π_t is not singular at the light-cone.

However, the smallness of the asymptotic masses might help formerly negligible contributions of higher-order diagrams to increase when approaching the light-cone. This may raise the necessity to reorganize the perturbation series.

In gauge theories one might expect that giving masses to the transverse gauge boson modes without modifying also their vertices might violate gauge invariance. Nevertheless, we shall see presently that gauge invariance of the HTL effective action is maintained if the above procedure is followed to generalize hard thermal loops to soft lightlike momenta.

3 Purely gluonic QCD

In this section we shall first treat purely gluonic QCD, where $e^2 = g^2N$. The improved HTL diagrams are obtained almost exactly as usually, the only difference being that the gluon propagator for hard momenta $p \geq \Lambda$ is modified to take into account the asymptotic thermal mass for transverse gluons,

$$G_{\mu\nu}(P) = A_{\mu\nu}\Delta_m + (B_{\mu\nu} + \alpha D_{\mu\nu})\Delta_0 \tag{6}$$

$$\text{with} \qquad \Delta_m = \frac{1}{P^2 - m_\infty^2} \quad , \quad \Delta_0 = \frac{1}{P^2}$$

[3] By employing a nonstandard real-time formalism, this feature can be kept also for covariant gauges [20].

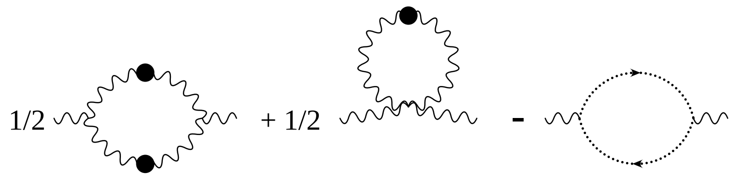

Fig. 2. The contributions to the improved gluon self-energy. The wavy lines with a blob represent gluon propagators whose transverse piece resums the asymptotic thermal mass; the dotted line represents the (unchanged) ghost propagator.

$$\text{and} \qquad A_{\mu\nu} = g_{\mu\nu} - B_{\mu\nu} - D_{\mu\nu} \quad , \quad B_{\mu\nu} = \frac{V_\mu V_\nu}{V^2} \quad , \quad D_{\mu\nu} = \frac{P_\mu P_\nu}{P^2} \tag{7}$$

where $V_\mu = U_\mu P^2 - P_\mu (PU)$ and $U_\mu = (1, \mathbf{0})$ is the four velocity of the thermal bath at rest.

3.1 Gluon self-energy

Since the collinear singularities in the bare one-loop polarization tensor are only logarithmic, the extraction of the T^2 contribution with soft external momenta can proceed as usually (see, however, the Appendix for a pitfall). Soft external momenta in the numerator can be neglected when compared to hard loop momenta, and the familiar expression is obtained, but with Δ_m in place of the originally massless propagators.

Let us demonstrate this in the example of $\Pi_{00}(Q)$. With the modified gluon propagator (6), the diagrams shown in Fig. 2 respectively yield

$$\begin{aligned} \Pi_{00}^{\rm loop}(Q) &= g^2 N \sum\!\!\!\!\!\!\!\int_P \left\{ \Delta_m - 4\Delta_m^- \Delta_m P_0 K_0 - \Delta_0^- \Delta_0 P_0 K_0 \right\} \\ \Pi_{00}^{\rm tad}(Q) &= -g^2 N \sum\!\!\!\!\!\!\!\int_P 3\Delta_m \\ \Pi_{00}^{\rm ghost}(Q) &= g^2 N \sum\!\!\!\!\!\!\!\int_P \Delta_0^- \Delta_0 P_0 K_0 \ , \end{aligned} \tag{8}$$

such that in the sum only massive propagators appear,

$$\Pi_{00}(Q) = g^2 N \sum\!\!\!\!\!\!\!\int_P \left\{ -2\Delta_m - 4\Delta_m^- \Delta_m P_0 K_0 \right\} \ . \tag{9}$$

In the following we use the imaginary-time formalism, so the zero components

of the momenta $P = (P_0, \mathbf{p})$ are discrete Matsubara frequencies $P_0 = 2\pi i n T$. The symbol $\sum\!\!\!\!\!\!\int_P$ is defined as

$$\sum\!\!\!\!\!\!\!\int_P = T \sum_n \int \frac{d^3p}{(2\pi)^3} \tag{10}$$

Throughout our paper Q is the external momentum, P is summed over and K is the difference $K = Q - P$. An index $^-$ means the transformation $P \to K$, e.g. $\Delta_0^- = 1/K^2$.

Performing the sum over the Matsubara frequencies yields $-2g^2 N \sum\!\!\!\!\!\!\int_P \Delta_m = \frac{3}{2} m^2$ with $m^2 = \frac{g^2 N T^2}{9}$ and

$$\begin{aligned}
4\sum\!\!\!\!\!\!\!\int_P \Delta_m^- \Delta_m P_0 K_0 \;=\; & \sum\!\!\!\!\!\!\!\int_P \left\{ \frac{1}{P_0 - \omega_p} + \frac{1}{P_0 + \omega_p} \right\} \left\{ \frac{1}{K_0 - \omega_k} + \frac{1}{K_0 + \omega_k} \right\} \\
= \int \frac{d^3p}{(2\pi)^3} & \left[n(\omega_p) + n(\omega_k) \right] \left(\frac{1}{Q_0 + \omega_p + \omega_k} - \frac{1}{Q_0 - \omega_p - \omega_k} \right) \\
& + \left[n(\omega_p) - n(\omega_k) \right] \left(\frac{1}{Q_0 + \omega_p - \omega_k} - \frac{1}{Q_0 - \omega_p + \omega_k} \right)
\end{aligned} \tag{11}$$

with $\omega_p^2 = p^2 + m_\infty^2$. After the frequency sum is done, we continue Q_0 to real values, keeping a small imaginary part $+i\varepsilon$ in mind.

Since $\omega_p - \omega_k$ is a soft quantity of order q,

$$\omega_p - \omega_k = zq - zq \frac{m_\infty^2}{2p^2} - \frac{q^2}{2p}(1 - z^2) + O\left(\frac{q^4}{p^3} \right), \quad z = \frac{\mathbf{pq}}{pq}, \tag{12}$$

it is useful to expand $n(\omega_k) = n(\omega_p) + n'(\omega_p)(\omega_k - \omega_p) + O(q^2)$. Moreover, the first two denominators on the r.h.s. of (11) are far from potential zeros for hard loop momenta, where Q_0 is negligible against $\omega_p + \omega_k$, so one can approximate $\omega_p, \omega_k \approx p$ there. The remaining denominators are those that give rise to collinear singularities at the light-cone when m_∞ is neglected. However, since these singularities are only logarithmic, we may also expand their numerators, dropping the terms that have too little power at large p to contribute to the T^2-part. This leads to

$$\Pi_{00} = 3m^2 + 2g^2 N \int \frac{d^3p}{(2\pi)^3} \, n'(p) \, \frac{Q_0^2}{Q_0^2 - (\omega_p - \omega_k)^2} \,. \tag{13}$$

Sufficiently far from $Q^2 = 0$, one can replace $\omega_p - \omega_k \to zq$. In the vicinity of the light-cone, however, the main contribution comes from $|z| \approx 1$ and

the second term in (12) becomes important, whereas the subsequent term is suppressed by $(1 - z^2)$. This latter term is effectively of $O(q^4/p^3)$, because the dominant contribution is generated when $(1 - z^2) \sim m^2/p^2$. We therefore obtain

$$\Pi_{00}(Q) = 3m^2 + 2g^2 N \int \frac{d^3p}{(2\pi)^3}\, n'(p)\, \frac{Q_0^2}{Q_0^2 + q^2 \frac{m_\infty^2}{p^2} - q^2 z^2} \tag{14}$$

$$= 3m^2 - \frac{g^2 N T^2}{3} \frac{3}{\pi^2} \int_0^\infty d\alpha \frac{\alpha^2 e^\alpha}{(e^\alpha - 1)^2} \frac{1}{2} \int_{-1}^{1} dz\, \frac{Q_0}{\sqrt{1 + \frac{\mu^2}{\alpha^2}}\, Q_0 - qz} \tag{15}$$

where $\mu^2 = \frac{m_\infty^2}{T^2} = g^2 N/6$.

Although the angular integration is no longer decoupled from the integral over the modulus of the loop momentum, this is still very close to the structure found in the conventional hard thermal loop, see Eq. (1). The difference is only that Q_μ does not appear in the scalar product with a light-like "unit vector" which is integrated over, but is contracted with a timelike one whose zero component is subject to a certain averaging.

Introducing

$$Y_\mu(\alpha) = (Y_0(\alpha), \mathbf{e}) \quad \text{with} \quad Y_0(\alpha) = \frac{\sqrt{\alpha^2 + \mu^2}}{\alpha} \tag{16}$$

and the double averaging

$$\Big\langle \quad \Big\rangle = \frac{3}{\pi^2} \int_0^\infty d\alpha \frac{\alpha^2 e^\alpha}{(e^\alpha - 1)^2} \frac{1}{4\pi} \int d\Omega \equiv \int_\alpha \int_\Omega \tag{17}$$

allows us to write the improved HTL polarization tensor in the same compact way as the conventional one,

$$\Pi_{\mu\nu} = 3m^2 \left\langle U_\mu U_\nu - \frac{UQ}{YQ} Y_\mu Y_\nu \right\rangle . \tag{18}$$

In this expression $Y_0 \neq 1$ is essential only in the denominator, since $\mu \sim g$. Strictly speaking, the lower bound on the integration variable α is given by Λ/T, but this is negligible when concentrating on the leading contributions $\sim g^2 T^2$.

$\Pi_{\mu\nu}$ is now defined also for soft lightlike momenta. The former singularity of

Π_{00} at $Q^2 = 0$ has disappeared and is replaced by $\ln(\text{const.}/g)$, to wit

$$\Pi_{00}(Q^2 = 0) = -\frac{g^2 N T^2}{3} \left\{ \ln \frac{2}{\mu} + \frac{1}{2} - \gamma + \frac{\zeta'(2)}{\zeta(2)} \right\} . \tag{19}$$

There is still a logarithmic branch cut for $|Q_0| < q$. The imaginary part along this cut, however, is a smooth albeit nonanalytic function in the momentum variables,

$$\Im m \Pi_{00}(Q) = \frac{9m^2}{2\pi} \frac{Q_0}{q} \int\limits_{\alpha_o(Q)}^{\infty} d\alpha \frac{\alpha^2 e^{\alpha}}{(e^{\alpha} - 1)^2} \quad \text{for } Q^2 < 0 \tag{20}$$

and with $\alpha_o(Q) = \mu Q_o / \sqrt{-Q^2}$. For $|Q^2|/Q_0^2 \gg g$, the lower integration bound $\alpha_o(Q)$ is of order g, hence negligible at leading order. However, it becomes of order 1 when $|Q^2|/Q_0^2 \sim g$, and finally goes up to infinity for $|Q^2| \to 0$, thereby bringing (20) down to zero.

3.2 Gluon vertex functions

For gluon vertex functions more complicated analytic structures arise. Performing the analogous steps as above, we find e.g. for the 000-component of the 3-gluon-vertex function

$$\begin{aligned}
&\Gamma_{000}(Q, R, -Q - R) \\
&= g^3 N \int \frac{d^3 p}{(2\pi)^3} n'(p) \left\{ R_0 \left(\frac{1}{Q_0 + \omega_{p-q} - \omega_p} \frac{1}{R_0 + \omega_{p-q-r} - \omega_{p-q}} \right. \right. \\
&\qquad \left. + \frac{1}{Q_0 - \omega_{p-q} + \omega_p} \frac{1}{R_0 - \omega_{p-q-r} + \omega_{p-q}} \right) \\
&\quad -(Q_0 + R_0) \left(\frac{1}{Q_0 + \omega_{p-q} - \omega_p} \frac{1}{Q_0 + R_0 + \omega_{p-q-r} - \omega_p} \right. \\
&\qquad \left. \left. + \frac{1}{Q_0 - \omega_{p-q} + \omega_p} \frac{1}{Q_0 + R_0 - \omega_{p-q-r} + \omega_p} \right) \right\}
\end{aligned} \tag{21}$$

after dropping all contributions $\ll g^3 T^2$.

Without the asymptotic thermal masses there are collinear singularities when any of the external momenta is light-like. These are again only logarithmic as long as the external momenta have different directions. Assuming this, one can for each denominator in turn expand the energy differences according to (12). This again amounts to modifying $Q_0 \to Q_0(1 + \frac{m_\infty^2}{2p^2})$ and likewise R_0.

After that the two pairs of terms in the round brackets can be combined by changing $\mathbf{p} \to -\mathbf{p}$.

As a result, the modified gluon vertex can be cast into the same form as was possible for the conventional HTL vertex

$$\Gamma_{000} = 3gm^2 \left\langle \frac{1}{(YQ)} \left\{ \frac{R_0}{(YR)} - \frac{Q_0 + R_0}{(Y(Q+R))} \right\} \right\rangle , \tag{22}$$

but with the averaging and the vector Y redefined according to (17) and (16).

Evaluating the angular integral gives

$$\Gamma_{000} = 3gm^2 \int_\alpha \left[R_0 \mathcal{M}(\bar{Q}, \bar{R}) - (Q_0 + R_0)\mathcal{M}(\bar{Q}, \bar{Q} + \bar{R}) \right] \tag{23}$$

where

$$\bar{Q} = (Q_0 \sqrt{1 + \mu^2/\alpha^2}, \mathbf{q}) \tag{24}$$

and $\mathcal{M}$ the Lorentz-invariant function introduced in Ref. [3],

$$\mathcal{M}(K,P) = \frac{1}{2\sqrt{-\Delta}} \ln \left(\frac{KP + \sqrt{-\Delta}}{KP - \sqrt{-\Delta}} \right) \quad , \quad \Delta = K^2 P^2 - (KP)^2 \tag{25}$$

In the conventional HTL result there are logarithmic singularities at $Q^2 = 0$, $R^2 = 0$, and $(Q + R)^2$ as well as (generally nonsingular) branch points at $\Delta(Q, R) \equiv \Delta(Q, Q + R) = 0$. In the improved result (22), these singularities are removed because in the α-average they are smeared according to (24).

As concerns higher vertex functions, the above argument that led to the form (22) can be essentially repeated.

3.3 Improved effective action

Because of the formal similarity of the improved hard thermal loops to the conventional ones — we only had to redefine the 4-vector that is used in the angular average and extend the averaging — it is natural to guess that one can also take over the compact effective action by Braaten and Pisarski [8]

$$\mathcal{S}_{\text{eff}} = -3m^2 \frac{1}{4} \int d^4x F_a^{\mu\alpha}(x) \left\langle \frac{Y_\alpha Y_\beta}{(YD)^2_{ab}} \right\rangle F_{b\ \mu}^{\beta}(x) \ , \tag{26}$$

but now with the definitions (16) and (17). These modifications obviously do not interfere with the manifest gauge invariance of the HTL effective action.

One can readily verify that (26) contains the improved version of the HTL two-point function, but some manipulations are needed to bring it into the form derived above (see the Appendix).

In the case of vertex functions, it is advantageous to use the earlier version of the HTL effective action of Taylor and Wong [6], which is, however, not manifestly gauge invariant. We therefore demonstrate that the redefinitions (16,17) do not interfere with its actual gauge invariance either.

We first write

$$\mathcal{S}_{\text{eff}} = \sum_{n=2}^{\infty} \mathcal{S}_n[A] \, , \tag{27}$$

where V_n collects the contributions n-linear in A_μ. From the improved gluon self-energy we have

$$\begin{aligned} \mathcal{S}_2[A] &= \frac{1}{2} \int \frac{d^4Q}{(2\pi)^4} A^{a\mu}(-Q)\, \Pi_{\mu\nu}(Q)\, A^{a\nu}(Q) \\ &= 3m^2 \int d^4x \, \mathrm{tr} \left\langle [H_0(x)]^2 - H(x) \frac{Y_0 \partial_0}{Y\partial} H(x) \right\rangle \end{aligned} \tag{28}$$

with the abbreviations $H(x) = Y^\mu A^a_\mu(x) \mathcal{T}^a$ and $H_0(x) = Y_0 A^a_0(x) \mathcal{T}^a$. With the ansatz

$$\mathcal{S} = 3m^2 \int d^4x \, \mathrm{tr} \left\{ \left\langle [H_0(x)]^2 \right\rangle - \langle \phi[H] \rangle \right\} \, . \tag{29}$$

one can find a gauge-invariant functional ϕ with "boundary condition" (28) as follows.

Under infinitesimal gauge transformations with parameter ω we have

$$\delta_\omega H_0 = Y_0 \partial_0 \omega - ig[H_0, \omega] \quad \text{and} \quad \delta_\omega H = Y\partial\omega - ig[H, \omega] =: \mathcal{D}\omega \, . \tag{30}$$

Gauge invariance of $\mathcal{S}$ implies $\delta_\omega \mathcal{S}[A] = 0$ for all ω. This requires

$$2 \langle Y_0 \partial_0 H \rangle = \langle \mathcal{D} \partial_H \phi[H] \rangle \, . \tag{31}$$

Up to terms that average to zero, a solution to this equation is given by a functional satisfying

$$\partial_H \phi[H] = \frac{2}{\mathcal{D}} Y_0 \partial_0 H \quad \text{with} \quad \frac{1}{\mathcal{D}} = \frac{1}{1 - \frac{1}{Y\partial}[igH, *]} \frac{1}{Y\partial} \, . \tag{32}$$

Note that here there is a small difference to the conventional case. Because of the new averaging procedure the inverse of $(Y\partial)$ always exists.

Counting only explicit coupling constants (not those implicit in Y_0 which depends on $\mu \propto g$), we have $g\partial_g \phi = \int d^4x \,\mathrm{tr}\, H \partial_H \phi$. That is

$$g\partial_g \phi[H] = 2 \int d^4x \,\mathrm{tr} \sum_{n=0}^{\infty} H \left\{ \frac{1}{Y\partial}[igH, *] \right\}^n \frac{1}{Y\partial} Y_0 \partial_0 H \, . \tag{33}$$

Integrating with respect to g yields

$$\mathcal{S}_{\text{eff}}[A] = 3m^2 \int d^4x \,\mathrm{tr} \left\langle (H_0)^2 + (Y_0\partial_0 H) F\left(\frac{1}{Y\partial}[iH, *] \right) \frac{1}{Y\partial} H \right\rangle \tag{34}$$
$$\text{where} \quad F(z) = 2 \sum_{n=0}^{\infty} \frac{z^n}{n+2} \, .$$

This functional is gauge invariant by construction. In fact, it is identical in form with that of Ref. [6], only the meaning of the symbols has changed. It reproduces the improved vertex functions in the form in which we had them obtained in the previous section, since Y_0's in the numerators can be put to 1. This just drops terms that are suppressed by powers of g, which we have always discarded.

We thus have shown that inclusion of the asymptotic gluon mass (5) removes the collinear singularities of the hard thermal loops without spoiling gauge invariance.

4 Inclusion of fermions

Defining the two structure functions of the fermion self-energy Σ at finite temperature by

$$\Sigma = aQ_0\gamma_0 + b\mathbf{q}\boldsymbol{\gamma} \quad , \quad a = \frac{1}{4Q_0} \,\mathrm{tr}\, \gamma_0 \Sigma \qquad b = -\frac{1}{4q^2} \,\mathrm{tr}\, \mathbf{q}\boldsymbol{\gamma}\Sigma \, , \tag{35}$$

the dressed propagator reads

$$S = -\frac{(1+a)Q_0\gamma_0 - (1+b)\mathbf{q}\boldsymbol{\gamma}}{(1+a)^2Q_0^2 - (1+b)^2q^2} \; . \tag{36}$$

The leading high-temperature contributions are

$$a = -\frac{M_f^2}{2qQ_0}\ln\left(\frac{Q_0+q}{Q_0-q}\right) \quad , \quad b = \frac{M_f^2}{q^2}\left\{1 - \frac{Q_0}{2q}\ln\left(\frac{Q_0+q}{Q_0-q}\right)\right\} \tag{37}$$

with $M_f^2 = \frac{1}{8}g^2C_fT^2$ (the QED case is covered by replacing $g^2C_f \to e^2$). For soft momenta, this contains an extra collective mode, the “plasmino”. However, as with the plasmon, its residue vanishes exponentially fast with increasing q. For $Q_0, q \gg gT$, the dressed propagator approaches

$$S(Q^2) = -\frac{Q_\mu\gamma^\mu + O(M^2/q)}{Q^2 - M_\infty^2 + O(M^4/q^2)} \; , \tag{38}$$

where

$$M_\infty^2 = 2(Q_0^2 a + q^2 b) = 2M_f^2. \tag{39}$$

Like the asymptotic gluon mass, this latter result (in contrast to (37)) is even an exact one-loop result when $Q^2 = 0$, i.e. not merely the leading high-temperature term for $Q_0, q \ll T$ (see the appendix of Ref. [22]).

With only external gluons the fermion loop produces the same HTL contributions as the purely gluonic one-loop diagrams did, except that instead of N there is an overall factor of $N_f/2$. For lightlike or nearly lightlike external momenta, the asymptotic fermion mass M_∞ becomes important and appears just in place of m_∞.

However, the situation is essentially different when hard thermal loops with external fermions are considered. The only hard thermal loops are those with two external fermion lines. Those are given by a loop which involves necessarily both internal fermion and gauge boson propagators, which have different asymptotic masses.

4.1 Fermion self-energy

Dressing the hard propagators in the fermion self-energy with their respective asymptotic masses gives the improved hard thermal loop

$$\Sigma(Q) = -\frac{1}{4} g^2 C_f \int \frac{d^3p}{(2\pi)^3} \left[n(p) + \tilde{n}(p)\right] \left(\gamma_0 - \frac{\mathbf{p}\boldsymbol{\gamma}}{p}\right)$$
$$\times \left\{ \frac{1}{Q_0 - \omega_p + \tilde{\omega}_{p-q}} + \frac{1}{Q_0 + \omega_p - \tilde{\omega}_{p-q}} \right\} \tag{40}$$

where $\tilde{n}$ is the Fermi-Dirac distribution function and $\tilde{\omega}_k = \sqrt{k^2 + M_\infty^2}$.

In the difference

$$\omega_p - \tilde{\omega}_k = zq + \frac{m_\infty^2 - M_\infty^2}{2p} - \frac{q^2}{2p}(1 - z^2) + O(\frac{q^3}{p^2}), \quad z = \frac{\mathbf{pq}}{pq}, \tag{41}$$

the dominant term that shifts the poles in the integrand of (40) is now independent of Q and proportional to the difference of the two asymptotic masses. The other correction in (41) that is of comparable magnitude is suppressed for $|z| \approx 1$, which makes it effectively of order q^3/p^2.

The two terms in the curly brackets of (40) therefore do not combine when exchanging $\mathbf{p} \to -\mathbf{p}$ and we arrive at an improved HTL fermion self-energy of the form

$$\Sigma(Q) = -M_f^2 \int_\alpha^f \int_\Omega \left\{ \frac{Y^\mu \gamma_\mu}{(YQ) + \frac{dm}{\alpha}} + \frac{Y^\mu \gamma_\mu}{(YQ) - \frac{dm}{\alpha}} \right\} \tag{42}$$

where

$$Y_\mu = Y_\mu(\infty) = (1, \mathbf{e}), \quad \int_\alpha^f = \frac{2}{\pi^2} \int_0^\infty d\alpha \frac{\alpha e^\alpha}{e^{2\alpha} - 1}, \quad dm = \frac{m_\infty^2 - M_\infty^2}{2T}. \tag{43}$$

So the 4-vector Y is light-like as with the conventional hard thermal loops. The collinear singularity is instead smeared out by the addition of $\pm dm/\alpha$ with a slightly modified averaging prescription for α.

The singularity of $\Re e \Sigma$ at the light-cone is again cut off by the asymptotic masses with the result

$$\Re e\, a(Q^2 = 0) = -\frac{M_f^2}{2q^2} \left\{ \ln \frac{q}{|dm|} + \frac{5}{4} \ln 2 + 1 - \gamma + \frac{\zeta'(2)}{\zeta(2)} \right\} \tag{44}$$
$$\Re e\, b(Q^2 = 0) = \Re e\, a(Q^2 = 0) + \frac{M_f^2}{q^2} \tag{45}$$

However, in contrast to the purely gluonic case, the imaginary part of the fermion self-energy is left unchanged for all spacelike momenta and remains

nonvanishing at the light-cone,

$$\Im m\, a(Q^2=0) = \Im m\, b(Q^2=0) = -\frac{\pi}{2}\frac{M_f^2}{q^2} \tag{46}$$

dropping rapidly but smoothly to zero only for timelike $Q^2 \gtrsim g^2(gT)^2$.

4.2 Improved effective action with fermions

In the case of the conventional hard thermal loops, the effective action for fermionic Green functions is given by the simplest possibility to bring its bilinear part, which is determined by the fermion self-energy, into a gauge-invariant form. With equal ease, this can be done with the improved HTL fermion self-energy,

$$\mathcal{L}_{\rm eff} = -\overline{\psi}(x) M_f^2 \int_\alpha^f \int_\Omega \left(\frac{Y^\mu}{iYD + \frac{dm}{\alpha}} + \frac{Y^\mu}{iYD - \frac{dm}{\alpha}} \right) \gamma_\mu \psi(x) \tag{47}$$

with $D^\mu = \partial^\mu \mathbf{1} - ig\mathcal{T}^a A^{a\mu}$.

This obviously requires that not only the fermion self-energy but all higher vertex functions can be broken up in two parts which differ only by the sign in front of dm. This is indeed the case as can be shown by induction.

Let us first explain how the structure of (42) arises. This simplest of the fermionic Green functions contains one fermionic and one bosonic propagator inside the loop. Decomposing each in partial fractions gives rise to a sum of products of the form

$$\Delta_F \Delta_B \to \sum_{\sigma_1\sigma_2} \frac{1}{F_0 + \sigma_1 \tilde{\omega}_f} \frac{1}{B_0 + \sigma_2 \omega_b} \tag{48}$$

where we have denoted the respective momenta by F_μ and B_μ and the σ_i are signs. F and B differ only by soft momenta, so we choose their orientation such that $F_\mu \approx +B_\mu$, and keep this convention when we shall be considering additional propagators in the loop. Performing now a further partial fractioning on (48) yields

$$\sum_{\sigma_1\sigma_2} \frac{1}{B_0 - F_0 + \sigma_1\omega_b - \sigma_2\tilde{\omega}_f} \left(\frac{1}{F_0 + \sigma_1\tilde{\omega}_f} - \frac{1}{B_0 + \sigma_2\omega_b} \right) . \tag{49}$$

Summing over the Matsubara frequencies will give $-\sigma_1 \tilde{n}(\omega_f)$ in the first, and $-\sigma_1 n(\omega_b)$ in the second term. Of the four parts of the sum only two can

contribute to hard thermal loops, namely those where $\tilde{n}(\omega_f)$ and $n(\omega_b)$ acquire the same sign, $\sigma_1 = \sigma_2$. Otherwise the contribution will be suppressed by a hard denominator in the prefactor.

Now let us consider a loop with one additional propagator, $\Delta_{F'}$ or $\Delta_{B'}$. This brings in another factor $1/(F_0' + \sigma'\tilde{\omega}_{f'})$ or $1/(B_0' + \sigma'\omega_{b'})$. Decomposing each product with the terms of (49) into partial fractions, one readily notices that soft denominators in the prefactors require like signs. Therefore, we still end up with exactly two contributions to the hard thermal loop, $\sigma' = \sigma_1 = \sigma_2 = \pm 1$. Moreover, of all the various combinations only those contribute to hard thermal loops where propagators of different statistics are combined so that in the end there is always a sum $\tilde{n} + n$ as in (40), for the difference in the arguments of $\tilde{n}$ and n does not matter.

The same argument applies to higher vertex functions, since in all the fermionic Green functions there is just enough power from hard loop momenta to produce a hard thermal loop $\propto T^2$ when all the energy denominators up to one are soft.

Usually these two contributions can be identified since only the first term of the r.h.s. of (41) is kept and this changes sign with the spatial loop integration variable. For the improved hard thermal loops, however, we have to keep also the subsequent term in (41), which does not.

Let us finally write down one example for an improved HTL vertex with external fermions. Either by direct calculation or by functional differentiation of (47) one finds for the quark-quark-gluon vertex

$$\begin{aligned}&\Gamma^\mu(Q,R;R-Q)\\ &= g\mathcal{T}\gamma_\nu M_f^2 \int_\alpha^f\!\!\int_\Omega \left\{ \frac{Y^\mu}{(YQ+\frac{dm}{\alpha})}\frac{Y^\nu}{(YR+\frac{dm}{\alpha})} + \frac{Y^\mu}{(YQ-\frac{dm}{\alpha})}\frac{Y^\nu}{(YR-\frac{dm}{\alpha})}\right\}\end{aligned} \tag{50}$$

Just as the analytic structure of the improved fermion self-energy turned out to be modified in a somewhat different fashion than was the case for the gluon self-energy, the vertex functions involve slightly different functions. Nevertheless, these can again be expressed in terms of integrals involving the function $\mathcal{M}$ introduced in (25). For instance,

$$\frac{1}{4}\,\mathrm{tr}\,\left(\gamma^0\Gamma^0\right) = g\mathcal{T}M_f^2\int_\alpha^f \left[\mathcal{M}(\tilde{Q}_+,\tilde{R}_+) + \mathcal{M}(\tilde{Q}_-,\tilde{R}_-)\right] \tag{51}$$

$$\text{with} \quad \tilde{Q}_+ = (Q_0 + \tfrac{dm}{\alpha}, \mathbf{q}) \quad \text{and} \quad \tilde{Q}_- = (Q_0 - \tfrac{dm}{\alpha}, \mathbf{q})\;. \tag{52}$$

Again the inclusion of the asymptotic mass (39) smoothes out the singularities in $\mathcal{M}$ in the final integration over α.

5 Conclusion

To summarize, dressing the hard propagators of the hard thermal loops by the asymptotic masses $\propto gT$ that pertain to the transverse branch of the gluonic excitations and to the normal branch of the fermionic ones removes all collinear singularities of the hard thermal loops and also preserves their gauge invariance. Moreover, the elegant effective actions of Braaten and Pisarski and of Taylor and Wong could be generalized to summarize the improved hard thermal loops in an equally compact form.

The gauge invariance of the improved hard thermal loops appears to be particularly encouraging to use them in place of the original hard thermal loops, where the latter lead to singular results when external lightlike momenta are involved. However, we have not yet shown that the systematics of a resummed perturbation theory built on the now everywhere well-defined hard thermal loops is as it was. It may well be that the would-be collinear singularities, which are only logarithmic at the level of hard thermal loops, build up in higher loop diagrams to a degree that overpowers the suppression by powers of g. Indeed, in Ref. [16] it has been found that in QCD the resummed one-loop diagrams are sufficiently singular to contribute to the sublogarithmic terms in eq. (19). Actually, the imaginary part (20) is even less stable and becomes large already for timelike momenta with $Q^2/q^2 \sim g$, thus preventing the longitudinal plasmon branch from coming arbitrarily close to the light-cone. This behaviour, which is completely opposite to the one observed in the case of scalar electrodynamics [14], will be the subject of a separate investigation [23].

Another place where collinear singularities do not cancel from a resummed calculation using ordinary hard thermal loops is the case of real soft photon production [13]. Using the improved hard thermal loops, on the other hand, gives a finite result when calculating the soft contribution at resummed one-loop order

$$E\frac{dW}{d^3p}\Big|_{\text{soft contr.}} \simeq \frac{Q_q^2\alpha\alpha_s}{2\pi^2}T^2\left(\frac{M_f}{E}\right)^2\ln\left(\frac{1}{g}\right)\ln\left(\frac{\Lambda}{M_f}\right) , \tag{53}$$

which coincides with the leading logarithms of Ref. [24].

However, there is also a hard contribution which has to restore independence of the scale Λ separating soft from hard momenta. For this it has to be such

that its mass singularities are also cut off in a way that produces a $\ln(1/g)$ besides a $\ln(T/\Lambda)$. Just like the internal hard propagators in the improved hard thermal loops this requires hard diagrams which are dressed by higher order diagrams. Again we expect that the main effect will be from the asymptotic thermal masses, which, as we have mentioned before, do not depend on a low-momentum limit (but they depend on Λ, which would become important in a more accurate calculation). So the same mechanism that renders finite the soft contribution should also apply to the hard one, and the last logarithm in (53) should combine into $\ln(T/M_f) \sim \ln(1/g)$.

Clearly, much work is still needed to first verify that an improved resummed perturbation theory really works as sketched here, and second to fully evaluate the sublogarithmic contributions. Encouraged above all by the gauge invariance of the improved hard thermal loops, we expect them to play a central role in this and analogous problems.

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Appendix

In order both to corroborate the results obtained in sect. 3.1 and to show a potential pitfall in their derivation, let us recalculate (15) by writing first

$$\Pi_{00}(Q) = -\frac{q^2}{Q^2}\Pi_\ell \quad \text{with} \quad \Pi_\ell = B^{\mu\nu}\Pi_{\mu\nu}. \tag{54}$$

(This assumes transversality of Π, which is guaranteed by the gauge invariance of the (improved) HTL effective action.)

The potential HTL contributions of $\Pi_{\mu\nu}$ give

$$\begin{aligned}\Pi_\ell(Q) = &-2e^2 \sum\!\!\!\!\!\!\!\int_P \Delta_m \\ &+4e^2 \sum\!\!\!\!\!\!\!\int_P \Delta_m \Delta_m^- \left[p^2 - \frac{(\mathbf{pq})^2}{q^2} + P^2 - \frac{(PQ)^2}{Q^2}\right] .\end{aligned} \tag{55}$$

This can be rewritten as

$$\Pi_\ell(Q) = \underbrace{4e^2 \sum\!\!\!\!\!\!\!\int_P \Delta_m \Delta_m^- \left[p^2 - \frac{(\mathbf{pq})^2}{q^2}\right]}_{(I)} + \underbrace{e^2(4m_\infty^2 - Q^2) \sum\!\!\!\!\!\!\!\int_P \Delta_m \Delta_m^-}_{(II)} \tag{56}$$

where, superficially, only the first part appears to be a hard thermal loop. It yields

$$\Pi_\ell^{(I)}(Q) = -\frac{Q^2}{q^2}\frac{e^2}{2\pi^2}\int_0^\infty dp\, p\, n(p) \int_{-1}^{1} dz \frac{z^2-1}{\left(z - Q_0/q\sqrt{1+m_\infty^2/p^2}\right)^2}. \tag{57}$$

The result (57) has exactly the form one obtains by expanding the manifestly gauge invariant effective action of eq. (26), which involves one more denominator than the one eq. (34). However, the derivative n' implicit in the averaging in (26) is replaced by $-2n$. For conventional hard thermal loops, this does not make any difference. But close to the lightcone it does. For example, the result (19) is not exactly reproduced— the $+\frac{1}{2}$ is missing.

What goes wrong here is that one has to divide Π_ℓ by Q^2 in order to obtain Π_{00}. This way it happens that, close to the light-cone, the second contribution in (56) can no longer be neglected. It reads

$$\Pi_\ell^{(II)}(Q) = (4m_\infty^2 - Q^2)\frac{e^2}{4\pi^2}\int_0^\infty dp\, \frac{n(p)}{p}\frac{Q^2}{Q_0^2(1+m_\infty^2/p^2) - q^2}. \tag{58}$$

Indeed, this by itself does not give rise to a HTL contribution, except when the Q^2 in the numerator is removed, which is done in (54). Then there is a linear singularity for $Q^2 \to 0$ that is cut off by m_∞^2/p^2 in the denominator, which heaves two powers of hard momentum into the numerator. This precisely accounts for the missing contribution to $\Pi_{00}(Q^2 = 0)$. Moreover, adding (58) to (57) can be shown to be equivalent to replacing n by $-\frac{1}{2}n'$ in the latter so that the gluon self-energy is exactly as prescribed by the manifestly gauge-invariant effective action (26).

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